

On the one-endedness of graphs of groups.

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Abstract

We give a technical result that implies a straightforward necessary and sufficient conditions for a graph of groups with virtually cyclic edge groups to be one ended. For arbitrary graphs of groups, we show that if their fundamental group is not one-ended, then we can blow up vertex groups to graphs of groups with simpler vertex and edge groups. As an application, we generalize a theorem of Swarup to decompositions of virtually free groups.

1 Introduction

A finitely generated group $G = \langle S \rangle$ is said to be *one-ended* if the corresponding Cayley graph $\text{Cay}(G, S)$ cannot be separated into two components by removing a finite subset. Otherwise G is said to be *many ended*. It is a classical result due to Stallings [Sta71] that a many-ended group either decomposes as an amalgamated free product, or an HNN extension, over a finite group.

Given the Bass-Serre correspondence between group actions on simplicial trees and their decompositions, or splittings, as (fundamental groups of) graphs of groups, c.f. [Ser03], a finitely generated group G is many ended if and only if it acts minimally, without inversions, and cocompactly on a simplicial tree T in which for some edge e the stabilizer G_e is finite.

It is often the case that a graph of groups with many ended vertex groups is itself one ended. As a prototypical example, the fundamental group of a closed surface is one ended but it is the amalgamated free product of free groups, which are many ended. In this paper we give necessary and sufficient conditions for a graph of groups to be one ended. The main result, Theorem 3.1, which is stated and proved in Section 3, is somewhat technical (although Figure 6 gives a good idea.) We do have some non immediate non-technical consequences, however.

We say that G is *one ended relative to a collection \mathcal{H} of subgroups* if for any G -tree T with finite edge stabilizers, there exists a subgroup $H \in \mathcal{H}$ that acts without a global fixed point. Or, equivalently, G is *many ended relative to \mathcal{H}* if G admits a non-trivial splitting as a graph of groups relative to \mathcal{H} (i.e. groups in \mathcal{H} are conjugate into vertex groups) with finite edge group.

Corollary 1.1. *If G_1 is one ended relative to a collection $\mathcal{H}_1 \cup \{C_1\}$, and G_2 is one ended relative to the $\mathcal{H}_2 \cup \{C_2\}$ with $C_1 \approx C_2$ virtually cyclic groups, then any free product with amalgamation of the form*

$$G_1 *_{C_1=C_2} G_2$$

is one ended relative to $\mathcal{H}_1 \cup \mathcal{H}_2$.

In the case of graphs of free groups with cyclic edge groups, this corollary (actually its natural generalization, c.f. Corollary 1.4) is proved in [Wil12, Theorem 18] and implied by results in [DF05]. Corollary 1.1 is false if we do not require the amalgamating subgroups to be virtually cyclic (or two ended). Nonetheless we can still understand the failure of one endedness of general graphs of groups.

Definition 1.2. A G -equivariant map $S \rightarrow T$ of simplicial G -trees is called a *collapse*, if T is obtained by identifying some edge orbits of S to points. In this case we also say that S is obtained from T by a *blow up*. We call the preimage \tilde{T}_v of a vertex v in S its *blowup*.

If a graph of groups with infinite edge groups is not one ended, then we can blow up some of its vertex groups. The following is formal consequence of Theorem 3.1.

Theorem 1.3. *If T is a G -tree (in which a collection of subgroups \mathcal{H} act elliptically) with infinite edge groups and G is not one-ended (relative to \mathcal{H}) then there is a vertex $v \in \text{Vertices}(T)$ and an edge $e \in \text{Edges}(T)$ with $v \in e$ such that the orbit of v can be blown up with G_v acting minimally on the non-trivial blowup \tilde{T}_v satisfying the following properties:*

- $G_e \leq G_v$ is the stabilizer of a vertex in \tilde{T}_v .
- The edge groups of \tilde{T}_v are conjugate in G_v to the vertex groups of an essential amalgamated free product or HNN decomposition of G_e with a finite edge group.

We note that blowing up a G -tree equivalent to *refining* a graphs of groups. If the vertex and edge groups of T are accessible à la Linnell or Dunwoody [Lin83, Dun85], then some of the new edge groups of the blowup are strictly smaller, in a certain sense. Theorem 1.3 therefore tells us that if a graph of groups is not one ended, then some vertex group can be compatibly decomposed as a graph of groups with “smaller” edge groups. We now use Theorem 1.3 to give a proof of Corollary 1.1:

proof of Corollary 1.1. We show the contrapositive. Let T be the Bass-Serre tree dual to the given splitting of G , and suppose that G is not one-ended relative to \mathcal{H} . Note that any decomposition of a virtually cyclic group as an HNN extension or an essential amalgamated free product must have finite edge groups. It follows that in all cases, by Theorem 1.3, some orbit of vertices Gv

can be blown up to minimal gG_vg^{-1} -trees with finite edge groups, implying that one of the vertex groups G_i fixing some vertex $v \in \text{Vertices}(T)$ acts minimally on \tilde{T}_v with finite edge stabilizers with

$$\mathcal{H}_i = \{H \in \mathcal{H} \mid H \cap G_i \neq \{1\}\}$$

and $C_i = G_e$ for some $v \in e \in \text{Edges}(T)$ acting elliptically. It follows that G_i is not one ended relative to $\mathcal{H}_i \cup \{C\}$. \square

This proof is easily adapted to give:

Corollary 1.4. *The fundamental group G of a graph groups with two-ended edge groups is one ended (relative to a collection \mathcal{H} of subgroups) if and only if every vertex group G_v is one ended relative to the incident edge groups (and the collection $\{H^g \cap G_v \mid g \in G, H \in \mathcal{H}\}$.)*

Using the full strength of Theorem 3.1, we will also generalize a result of Swarup on the decomposition of free groups [Swa86] to virtually free groups. This result was already partially generalized by Cashen [Cas12] to decomposition of virtually free groups with virtually cyclic edge groups.

Definition 1.5. We write $H \leqslant G$ to signify that G splits essentially as a graph of groups with finite edge groups and H is a vertex group.

Theorem 1.6. *Let G be finitely generated and virtually free.*

1. *If G splits as an amalgamated free product $G = A *_C B$ with C finitely generated and infinite then without loss of generality, interchanging A and B if necessary, there is some infinite subgroup $C_1 \leqslant C$ such that $C_1 \leqslant A$.*
2. *If G splits as an HNN extension $G = A *_C t$ with C finitely generated and infinite, then there is an infinite subgroup $C_1 \leqslant C$ and a splitting of A as a graph of groups with finite edge groups such that without loss of generality, interchanging C and $t^{-1}Ct$ if necessary, C_1 is a vertex group of this splitting of A and $t^{-1}C_1t$ is conjugate in A into another vertex group of this splitting.*

Unlike in Swarup's proof, we do not use homological methods. Our proof is more along the lines of the geometric arguments found in [Wil12, Lou08, BF93, DF05]. The main difference is that instead of working with graphs of spaces X with $\pi_1(X) = G$, we consider a more abstract object, namely the G -cocompact core of the product of two G -trees [Gui05]. This gives a nice way to handle torsion, which would otherwise give “messy” graphs of spaces. The core has been used before to study pairs of group splittings, in particular, Fujiwara and Papasoglu in [FP06] use it to show the existence of QH subgroups for one ended groups that have hyperbolic-hyperbolic pairs of slender splittings. Although it could be noted that the action of our group on the core gives rise to a G -orbihedron à la [Hae91], we will not need this machinery; in fact, modulo classical Bass-Serre theory and Guirardel's Core Theorem for simplicial trees, Theorem 2.3 (of which we sketch a proof), our argument is self-contained.

1.1 Acknowledgements

I wish to thank John MacKay and Alessandro Sisto for asking me for a proof of Corollary 1.1. I had actually even given them what I thought to be a counterexample; their counterexample gave me ample motivation to investigate this problem further. This paper also would not have been possible without everything I learned from Lars Louder. The ideas of Section 2, especially the usefulness of the Core Theorem, arose from our discussions while working on strong accessibility. Finally, I wish to thank Inna Bumagin, the quasi entirety of the work for this paper was done while I was supported by her NSERC grant.

2 Preliminaries

2.1 Group actions

All group action will be from the left. Let X be a G -set, if $S \subset X$ is a subset, we will denote by G_S the set $\{g \in G \mid gS = S\}$. If $S = \{x\}$ is a singleton, then we will write G_x instead of $G_{\{x\}}$. We call a subset $S \subset X$ G -regular if for any $x, y \in S$ in the same G -orbit there is some $g \in G_S$ such that $gx = y$. The following lemma is immediate:

Lemma 2.1. *Let X be a G -set. If $S \subset X$ is G -regular, then we have an embedding*

$$G_S \backslash S \hookrightarrow G \backslash X.$$

In this paper, all trees will be simplicial. In particular we will consider them to be topological spaces, equipped with a CW-structure which also makes them into graphs. We further metrize these graphs by viewing edges as real intervals of length 1. We assume the reader is familiar with Bass-Serre theory and we will switch freely between G -trees and splittings as graphs of groups, viewing the two as being equivalent.

Let T be a G -tree. T is *essential* if every edge of T divides it into two infinite components. We say a G -tree T is *without inversions* if, for any edge $e \in \text{Edges}(T)$, if $ge = e$ then g fixes e pointwise. Equivalently, if $u, v \in \text{Vertices}(T)$ are the vertices at the ends of the edge e , then we have inclusions

$$G_u \geq G_e \leq G_v.$$

We call vertex and edge stabilizers, vertex groups and edge groups respectively. We say that T is *minimal* if there are no proper subtrees $S \subset T$ with $G_S = G$. We say T is *cocompact* if $G \backslash T$ is compact. An element g or a subgroup H of G are said to *act elliptically on T* if the groups $\langle g \rangle$ or H fix some $v \in \text{Vertices}(T)$.

2.2 Products of trees, cores, and leaf spaces

Suppose that T_1 and T_2 are G -trees, then we have a natural induced action $G \curvearrowright T_1 \times T_2$. Since we consider the trees T_1, T_2 as 1 dimensional CW complexes, we

may consider their product $T_1 \times T_2$ as a *square complex*, that is a 2 dimensional CW complex whose cells consist of vertices, edges, and squares. There are natural projections $p_i : T_1 \times T_2 \rightarrow T_i$. The following Lemma is immediate:

Lemma 2.2. *If the actions $G \curvearrowright T_1$ and $G \curvearrowright T_2$ are without inversions, then so is the action $G \curvearrowright T_1 \times T_2$, i.e. if $\sigma \supset \epsilon$ is an inclusion of cells (e.g. a square containing an edge), then $G_\sigma \leq G_\epsilon$.*

If the collection of subgroups \mathcal{H} act elliptically on T_1 and T_2 then each subgroup of \mathcal{H} fixes a vertex of $T_1 \times T_2$.

The action $G \curvearrowright T_1 \times T_2$ is not cocompact in general. It turns out, however, that we can extract a useful subset, namely Guirardel's convex core.

Theorem 2.3 (The Core Theorem [Gui05, c.f. Théorème principal and Corollaire 8.2]). *Let $G \curvearrowright T_1$, $G \curvearrowright T_2$ be two minimal actions of a finitely generated group G on simplicial trees T_1, T_2 with finitely generated edge stabilizers. Then there is a G -invariant subset $\mathcal{C} \subset T_1 \times T_2$ called the core of the action $G \curvearrowright T_1 \times T_2$ which is defined as the smallest connected G -invariant subset such that the restrictions of the projections $p_i|_{\mathcal{C}} : \mathcal{C} \rightarrow T_i$ have connected fibres. The quotient $\mathcal{S} = G \backslash \mathcal{C}$ is compact.*

The restrictions of the projections $p_i|_{\sigma} : \sigma \rightarrow T_i$ are well defined for each cell (i.e. a vertex, edge, square) $\sigma \subset T_1 \times T_2$. If σ is a square then the projection is onto an edge $p_i(\sigma) \in \text{Edges}(T)_i$. If $\lambda_1, \lambda_2 \subset \sigma$ are two fibers of such a projection, see Figure 1, we can define a distance $d_i^\sigma(\lambda_1, \lambda_2)$ to be the distance in $p_i(\sigma)$

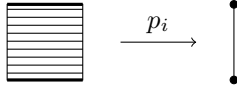


Figure 1: The projection of a square on an edge and some of its fibers

between the points $p_i(\lambda_1)$ $p_i(\lambda_2)$; thus putting a metric d_i^σ on the set of p_i -fibers in a cell σ . We now define the *i-leaf space* \mathcal{L}_i of a subset $Z \subset T_1 \times T_2$ to be set of connected unions of p_i -fibers of cells in Z , called *leaves*, so that we see Z as being *foliated* by the leaves in \mathcal{L}_i . We metrize \mathcal{L}_i with a metric d_i that agrees with all the d_i^σ , but which maximizes distances

As a consequence of the direct product structure we have the following.

Lemma 2.4. *Let $Z \subset T_1 \times T_2$, then the leaf spaces \mathcal{L}_i are forests (see Figure 2).*

If $\mathcal{C} \subset T_1 \times T_2$ is a core then the leaf spaces \mathcal{L}_i are homeomorphic to the trees T_i . Later, however, we will be performing operations that will alter the leaf spaces.

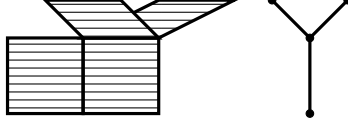


Figure 2: The i -leaves in a square complex and the resulting leaf space, which is a tree.

2.3 Induced splittings

Let $v \in \text{Vertices}(T_i)$, $e \in \text{Edges}(T_i)$ and let m_e be the midpoint of e . Let $\tau_v = p_i^{-1}(\{v\}) \cap \mathcal{C}$ and $\tau_e = p_i^{-1}(\{e\}) \cap \mathcal{C}$. By Theorem 2.3 the preimages τ_v, τ_e are connected and are therefore leaves in \mathcal{L}_i .

Since we have an action $G \curvearrowright \mathcal{C}$, since τ_v, τ_e are defined as T_i -point preimages, and since G_v, G_e are exactly the stabilizers of these points v, m_e , the subsets $\tau_v, \tau_e \leq \mathcal{C}$ are G -regular so by Lemma 2.1 we have embeddings

$$G_v \backslash \tau_v \hookrightarrow G \backslash \mathcal{C} \hookleftarrow G_e \backslash \tau_e.$$

By Theorem 2.3, $G \backslash \mathcal{C}$ is compact therefore the quotients $G_v \backslash \tau_v, G_e \backslash \tau_e$ must be as well. Moreover, because τ_v, τ_e are contained in p_i -fibres, for $j \neq i$ the restrictions

$$p_j|_{\tau_v} : \tau_v \rightarrow T_j, p_j|_{\tau_e} : \tau_e \rightarrow T_j$$

are injective. Finally, the projection $p_j|_{\mathcal{C}} : \mathcal{C} \rightarrow T_j$ is G -equivariant; we have shown the following.

Lemma 2.5. *Let $v \in \text{Vertices}(T_i)$, $e \in \text{Edges}(T_i)$, $j \neq i$, then the fibres τ_v, τ_e are mapped via injectively p_j to subtrees that are G_v, G_e -invariant (respectively). Viewed as subsets of the core $\mathcal{C} \subset T_1 \times T_2$, τ_v and τ_e coincide with their j -leaf spaces.*

The actions $G_e \curvearrowright \tau_e, G_v \curvearrowright \tau_v$ are cocompact, moreover they are essential if and only if the actions of the subgroups $G_v \curvearrowright T_j, G_e \curvearrowright T_j$ are without global fixed points.

The G_v, G_e -trees τ_v, τ_e give splittings induced by the action on T_j . The blowups of Theorem 3.1 will be obtained by modifying the trees τ_v . For aficionados of CAT(0) cube complexes, it is worth remarking that the core \mathcal{C} is a CAT(0) square complex, in fact a \mathcal{VH} -complex, and that the set of fibres $\tau_e, e \in \text{Edges}(T_i)$ is the set of hyperplanes.

2.4 Spurs, free faces, and cleavings

In the previous section we obtained cocompact G_v, G_e -trees τ_v, τ_e . We say a tree has a *spur* if it has a vertex of degree 1, an edge adjacent to a spur is called a *hair*. We now give a shaving process.

Lemma 2.6. *Let T be a cocompact G -tree. T is minimal if T doesn't have any spurs. If T is not minimal, then we can obtain the minimal subtree $T(G)$ as the final term of a finite sequence*

$$T = T_0, \dots, T_k = T(G),$$

where T_{i+1} is obtained from T_i by G -equivariantly contracting one G -orbit of hairs to points.

Proof. Let $v \in \text{Vertices}(T)$ be a spur adjacent to an edge $e \in \text{Edges}(T)$ and let $u \in \text{Vertices}(T)$ be the other endpoint of e . The map $T \rightarrow T$ obtained by G -equivariantly collapsing ge onto gu ; $g \in G$ is a deformation retraction onto a proper G invariant subtree, so T is not minimal.

Suppose now that T is not minimal then there is some proper G -invariant subtree $S \subset T$. Let K be the closure of some connected component of $T \setminus S$. Then $K \cap S\{v\}$ for some $v \in \text{Vertices}(S)$. Since S is G -invariant and connected, we must have $G_K \leq G_v$. It follows that for any $w \in \text{Vertices}(K)$ and any $g \in G_k$ the distance $d_T(w, v) = d_T(gw, v)$, i.e. the action of G_K on K is the action on a rooted tree with root v . Since K is G -regular, we have an embedding $G_K \backslash K \hookrightarrow G \backslash T$ which is compact; thus K must have finite radius since G_K preserves distances from the root.

Since K is a tree with finite diameter it must have vertices of valency 1. By the argument at the beginning of the proof we can G_K -equivariantly collapse hairs and since $G_K \curvearrowright K$ is cocompact, after finitely many collapses we will have collapsed K to v . Again since $G \curvearrowright T$ is cocompact, there are only finitely many orbits of connected components of $T \setminus S$, so the result follows. \square

Let σ be a square in some $Z \subset T_1 \times T_2$ we say an edge $\epsilon \subset \sigma$ is a *free-face*. The following terminology is due to Wise [Wis04].

Definition 2.7. Let $e \in \text{Edges}(T_i)$ and let $\tau_e \subset \mathcal{C}$ be the fibre mentioned in Lemma 2.5, the *hypercarrier* $\mathcal{H}(\tau_e)$ is the union of squares containing τ_e .

We note that for $e \in T_i$, a hypercarrier is mapped to an edge of T_i and that $\mathcal{H}(\tau_e)$ is homeomorphic to $\tau_e \times [-1, 1]$.

Definition 2.8. We say an edge ϵ in some $Z \subset T_1 \times T_2$ is *i-transverse* if it coincides with its *i*-leaf space, or equivalently it is mapped monomorphically via $p_i|_\epsilon$, or equivalently if it is contained in a *j*-leaf.

An immediate consequence of Lemma 2.6 and Figure 3 is the following.

Lemma 2.9. *Let $e \in \text{Edges}(T_i)$, if $G_e \curvearrowright \tau_e$ is not minimal then $\mathcal{H}(\tau_e)$ has a square σ containing an *i-transverse free face* ϵ .*

We now borrow some terminology from [DF05].

Definition 2.10. A simplicial map $S \rightarrow T$ between two trees that is obtained by identifying edges sharing a common vertex is called a *folding*. If T is obtained from S by a folding, then we say S is obtained from T by a *cleaving*.

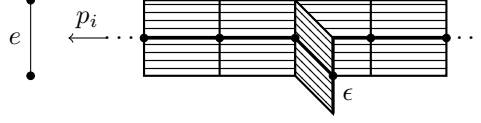


Figure 3: A spur of τ_e and the corresponding free face ϵ in the hypercarrier $\mathcal{H}(\tau_e)$.

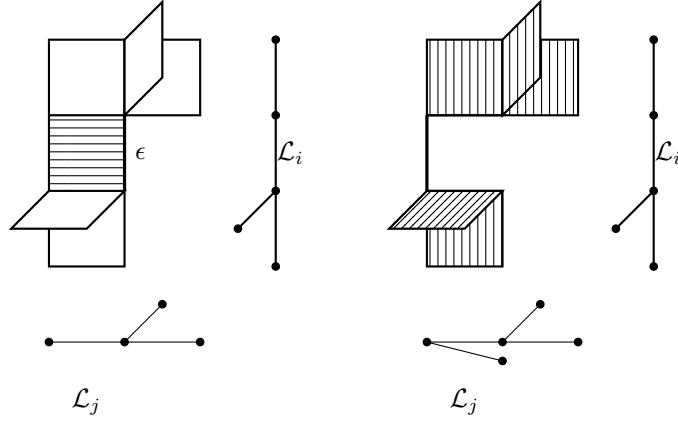


Figure 4: The effects of collapsing an i -transverse free face ϵ : the leaf space \mathcal{L}_j gets cleaved, \mathcal{L}_i remains unchanged. On the right the j -leaves are drawn.

We now have the following which is immediate (see Figure 4).

Lemma 2.11. *Let $\epsilon \subset Z \subset T_1 \times T_2$ be an i -transverse free edge in a square σ . If we collapse σ onto the face opposite to ϵ the leaf space \mathcal{L}_i is unchanged and the leaf space \mathcal{L}_j gets cleaved.*

In fact this lemma can be used backwards to give a proof of Theorem 2.3. We will sketch it, leaving the details to an interested reader familiar with folding sequences [BF91, Sta91, Dun98, KWM05].

sketch of the proof of Theorem 2.3. Pick some vertex $v \in T_1 \times T_2$ and consider its G -orbit. We can add finitely many connected G -orbits of edges to get a connected G -complex $Gv \subset \mathcal{C}_1 \subset T_1 \times T_2$. \mathcal{C}_1 has leaf spaces $\mathcal{L}_1, \mathcal{L}_2$ which project onto T_1, T_2 . The disconnectedness of the fibres of the projections $p_i|_{\mathcal{C}_1} : \mathcal{C}_1 \rightarrow T_i$ coincides with the failure of injectivity of the projections $\mathcal{L}_i \rightarrow T_i$. By Lemma 2.11 (backwards) adding a square can give a folding of one of the leaf spaces. Since the edge groups of T_1, T_2 are finitely generated, and because adding all the squares of $T_1 \times T_2$ folds \mathcal{L}_i to T_i , it follows that the leaf spaces \mathcal{L}_i can be made to coincide with T_i after adding finitely many G -orbits of squares. \square

3 The statement and proof of the main theorem

For this section we fix a collection \mathcal{H} of subgroups of G we let $T_\infty, T_{\mathcal{F}}$ be cocompact, minimal G -trees in which the subgroups in \mathcal{H} act elliptically. We further require that edge groups of T_∞ are infinite and finitely generated and edge groups of $T_{\mathcal{F}}$ are finite.

Theorem 3.1 (Main Theorem). *Let \mathcal{H} be a collection of subgroups of G and let $T_\infty, T_{\mathcal{F}}$ be cocompact, minimal G -trees in which the subgroups in \mathcal{H} act elliptically, suppose furthermore that the edge groups of $T_{\mathcal{F}}$ are finite. Then there exists a vertex $v \in \text{Vertices}(T_\infty)$ and a cocompact, minimal G_v -tree \check{T}_v such that*

- (i) *for every $f \in \text{Edges}(T_\infty)$ incident to v the subgroups $G_f \leq G_v$ act elliptically on \check{T}_v . Also for every $H \in \mathcal{H}, g \in G$ the subgroup $H^g \cap G_v \leq G_v$ acts elliptically on \check{T}_v .*

Moreover,

- (1) *either every edge group of \check{T}_v is finite; or*
- (2) *there is some edge $e \in T_\infty$ that is incident to v such that*
 - (a) *G_e splits essentially as an amalgamated free product or an HNN extension with finite edge group.*
 - (b) *G_e is the stabilizer of some vertex of \check{T}_v , and*
 - (c) *the edge stabilizers of \check{T}_v are conjugate in G_v to the vertex group(s) of the splitting of G_e found in (2a).*
 - (d) *The vertex groups of \check{T}_v that are not conjugate in G_v to G_e are also vertex groups of a one edge splitting of G_v with a finite edge group.*

An example of what happens in situation (2) is shown in Figure 6.

Proof. Let \mathcal{C} be the core of $T_\infty \times T_{\mathcal{F}}$, the ∞ -leaf space \mathcal{L}_∞ is the tree T_∞ , and we can see \mathcal{C} as a tree of spaces (c.f. [SW79] for details) with vertex spaces the trees $\tau_v; v \in \text{Vertices}(T_\infty)$ and edge spaces $\mathcal{H}(\tau_e) = \tau_e \times [-1, 1]; e \in \text{Edges}(T_\infty)$ attached to the τ_v along the subspaces $\tau_e \times \{\pm 1\}$.

It may be that for some $e \in \text{Edges}(T_\infty)$ the G_e -trees τ_e are not minimal. By Lemmas 2.9, 2.6, 2.11, we can repeatedly G -equivariantly collapse ∞ -transverse free faces, so that after finitely many steps we obtain a “shaved” core \mathcal{C}'_s such that $\tau_e \cap \mathcal{C}_s$, which we will still denote τ_e , are minimal G_e trees. Although the \mathcal{F} -leaf space got cleaved repeatedly, the ∞ -leaf space is unchanged, we still write $\mathcal{L}_\infty = T_\infty$.

Now if T is a G -tree and $H \leq G$, denoting by $T(S)$ the minimal S -invariant subtree, we have $T(H) \subset T(G)$. It therefore follows, by Lemma 2.6, after finitely many equivariant spur collapses we can make the G_v -trees τ_v minimal, without affecting the previously shaved attached hypercarriers $\mathcal{H}(\tau_e)$, while preserving the leaf space $\mathcal{L}_\infty = T_\infty$. We call the resulting complex $\mathcal{C}_s \subset \mathcal{C}'_s \subset \mathcal{C}$, where the trees τ_v and τ_e are minimal, the ∞ -minimal core.

For every edge $k \in \text{Edges}(T_{\mathcal{F}})$, G_k is finite, therefore a minimal G_k tree is a point; thus, by cocompacity and regularity, the trees $\tau_k \in \mathcal{C}$ have finite diameter and the same must be true of every connected component of $\tau_k \cap \mathcal{C}_s$, so every connected component of $\tau_k \cap \mathcal{C}_s$ has a spur. It therefore follows that \mathcal{C}_s must have an \mathcal{F} -transverse free edge ϵ containing a spur of some connected component of $\tau_k \cap \mathcal{C}_s$ for some $k \in \text{Edges}(T_{\mathcal{F}})$. Furthermore the stabilizer $G_{\epsilon} \leq G_{p_{\mathcal{F}}(\epsilon)}$ which is an edge stabilizer of $T_{\mathcal{F}}$ and therefore is finite.

Definition 3.2. Let T be a minimal G -tree and let $e \in \text{Edges}(T)$. We denote by $C(T, e)$, the *non- e -collapse of T* , the tree whose edges are the edges in the orbit $Ge \subset T$ and whose vertices are the closures of the connected components of $T \setminus Ge$, with $v \in \text{Vertices}(C(T, e))$ adjacent to $e \in \text{Edges}(C(T, e))$ if and only if, viewed as subsets of T , $e \cap v \neq \emptyset$.

This \mathcal{F} -transverse free edge ϵ must be contained in some τ_v ; $v \in \text{Vertices}(T_{\infty})$. Suppose first that ϵ was not contained in any hypercarrier attached to τ_v , then for every $e \ni v$ in $\text{Edges}(T_{\infty})$, G_e fixes some hypercarrier $\mathcal{H}(\tau_e)$ such that $\mathcal{H}(\tau_e) \cap \tau_v = \tau_e^+$ is contained in the complement $\tau_v \setminus G_v \epsilon$. It therefore follows that $\tilde{T}_v = C(\tau_v, \epsilon)$ is a tree with finite edge groups, in which each $G_e \leq G_v$, $e \in \text{Edges}(T_{\infty})$ act elliptically, and also conjugates of groups in \mathcal{H} intersecting G_v act elliptically; thus (i) and (1) are satisfied.

Otherwise the free edge $\epsilon \subset \tau_v$ is, by definition of a free edge, contained in *exactly one* hypercarrier $\mathcal{H}(\tau_e)$. We will now construct the G_v -tree \tilde{T}_v satisfying the criteria of (2), this construction is illustrated in Figure 5. We first take the

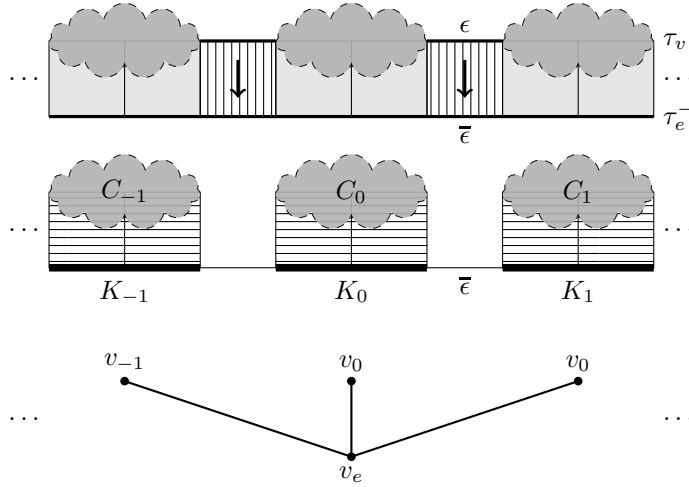


Figure 5: Constructing \tilde{T}_v . The top shows a portion of Z_c , the middle shows the result of equivariantly collapsing the free face ϵ , the bottom shows the corresponding ∞ -leaf space.

subset

$$Z = \left(\tau_v \bigcup_{e \ni v} \mathcal{H}(\tau_e) \right) \cap \mathcal{C}_s,$$

i.e. τ_v with all the adjacent hypercarriers. The ∞ -leaf space for Z is the union of set of edges of T_∞ connected to v . Now the G_v translates of ϵ are contained in the hypercarriers $\mathcal{H}(\tau_{ge}); g \in G_v$. For each such hypercarrier we denote by τ_{ge}^- the connected component of $\mathcal{H}(\tau_{ge})$ not contained in τ_v (see the top of Figure 5.)

We now G_v -equivariantly collapse the square $\sigma \supset \epsilon$ onto opposite the edge $\bar{\epsilon}$, to obtain a G_v -subset $Z_c \subset Z$ (see the middle of Figure 5.) The resulting intersection $\tau_v \cap Z_c$ consists of a collection of connected components $\{C_i \mid i \in I\}$. Similarly, the G_e -translates of $\bar{\epsilon}$ give connected components $\{K_i \mid i \in I\}$ of $\tau_e \setminus G_e \bar{\epsilon}$. Because G_e acts on $C(\tau_e^-, \bar{\epsilon})$, and by minimality of τ_e , this action is also minimal with one edge orbit. This gives us (2a).

For every edge $v \in f \in \text{Edges}(T_\infty)$ that is not in the G_v -orbit in e , the orbit $G_v \epsilon$ did not intersect $\mathcal{H}(\tau_f) \cap \tau_v$. It follows that each such $G_f \leq G_v$ stabilizes some component C_i .

We now delete from Z_c all hypercarriers not stabilized by a G_v -conjugate of G_e to obtain a G_v complex $Z'_c \subset Z_c$. Next we collapse each translate of τ_e^- to a vertex v_e , collapse each component C_i to a vertex v_i , and collapse each connected component of G_v -translates of $\mathcal{H}(\tau_e) \cap Z'_c$ onto an edge; or, equivalently, we take the ∞ -leaf space of Z'_c we obtain a G_v -tree \check{T}_v that satisfies (2b) and (i). See the bottom of Figure 5. We moreover note that, by construction, every subgroup $H^g \cap G_v; g \in G, H \in \mathcal{H}$ acts elliptically on \check{T}_v .

The vertex stabilizers of $C(\tau_e^-, \bar{\epsilon})$ coincide with the component stabilizers $(G_e)_{K_i} = (G_v)_{C_i}$; since τ_e^- is G_v -regular. We also have $(G_v)_{C_i} \cap (G_v)_{\tau_e^-} = (G_v)_{K_i}$ (again see the middle of Figure 5) it follows that the edges stabilizers of \check{T}_v satisfy (2c).

Finally note that the vertex groups of \check{T}_v that are not stabilized by G_v -conjugates of G_e are also the vertex groups of $C(\tau_v, \epsilon)$ (see the top of Figure 5), since G_e is finite (2d) follows. \square

4 Splittings of virtually free groups

Another way to use Theorem 3.1 is to obtain cleavings of G -trees whose edge and vertex groups are “smaller”. This will be used as the inductive step in our proof of Theorem 1.6.

Corollary 4.1. *Let T be a G -tree in which the subgroups \mathcal{H} act elliptically with infinite edge groups and let G be many ended relative to \mathcal{H} . Either some vertex $v \in \text{Vertices}(T)$ can be blown up to a tree with finite edge groups; or there is edge $e \in \text{Edges}(T)$ such that we can blow up T to some tree \check{T} , and then collapse the edges in the orbit of e to points. The resulting tree T' is obtained from T by equivariantly cleaving some edge e . If $e' \in \text{Edges}(T')$ is a new edge obtained by*

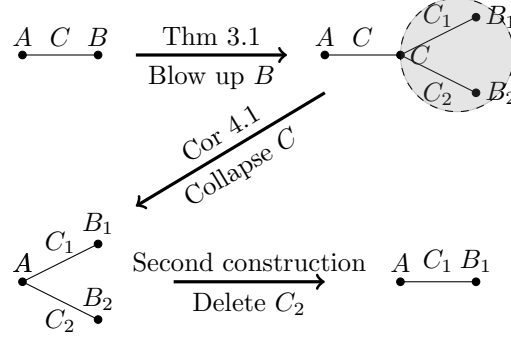


Figure 6: An example of the effects of Theorem 3.1, Corollary 4.1, and the second construction of the proof of Theorem 1.6, on a graph of groups. The vertices and edges are labelled by the corresponding vertex and edge groups. In all cases $B_i < B$ and $C_i < C$.

a cleaving of e then $G_{e'} < G_e$. Also for each new vertex $v' \in \text{Vertices}(T')$ there is some $v \in \text{Vertices}(T)$ that got cleaved such that $G_{v'} < G_v$.

Furthermore, in passing from T to T' the number of edge orbits and the number of vertex orbits does not decrease and increases by at most 1.

Proof. Suppose we are in case (2) of Theorem 3.1, then some vertex v gets blown up to \tilde{T}_v and some vertex stabilizer of \tilde{T}_v coincides with G_e . It follows that in the blown up tree \tilde{T} , the edge e stabilized by G_e has one endpoint w whose stabilizer G_w coincides with G_e . If we collapse this edge we therefore do not enlarge any of the vertex or edge groups. The required properties of T' are immediately satisfied by Theorem 3.1 (see Figure 6.) \square

Finally, we can give our description of the decompositions of virtually free groups as amalgamated free products or HNN extensions.

proof of Theorem 1.6. We shall prove this result by successively applying Corollary 4.1 until some desirable terminating condition is met. On one hand, virtually free groups have no one-ended subgroups so we will always be able to apply our Corollary; on the other hand, virtually free groups being quasi isometric to free groups are Gromov-hyperbolic; thus finitely presented and having only finitely many conjugacy classes of finite subgroups. It therefore follows by either Linnell or Dunwoody accessibility [Dun85, Lin83] there are no infinite chains $C_1 > C_2 > \dots$ of virtually free groups. Before starting we have a couple constructions.

First construction (pass to relatively one ended vertex subgroups): Let T be a G tree with one edge orbit Ge with G_e infinite. By accessibility, we may pass to a tree $T^{(2)}$ obtained by blowing up some vertices v of T to trees \tilde{T}_v such that the vertex groups of \tilde{T}_v are one ended relative to the stabilizers G_f

of the incident edges $f \ni v$. If possible, we take $T^{(1)} \subset T^{(2)}$ to be an infinite connected subtree obtained by deleting edges with finite stabilizers and we set $G^{(1)} = G_{T^{(1)}}$. We note that the vertex groups of $T^{(1)}$ are \leq the vertex groups of T , and vertex groups are one ended relative to the incident edge groups.

Second construction (pass to smaller edge groups): The second construction utilizes Corollary 4.1. If T_i is a G_i -tree with one edge orbit whose vertex groups are one ended relative to the incident edge groups, we first apply Theorem 3.1 to blow up a vertex $v \in \text{Vertices}(T_i)$, and find ourselves in case (2) of the Theorem. If \tilde{T}_v has a finite edge group then G_v is not one-ended relative to the incident edge groups, contradicting our assumption. By Corollary 4.1 we can collapse an edge of the blowup of T_i to get a cleaving T'_i that has at most two edge orbits, with edge groups $<$ the edge groups of T_i . The new vertex groups are also \leq the old vertex groups. If there are two edge orbits we obtain $T_{i+1} \subset T'_i$ as an infinite subtree containing only one edge orbit and we set $G_{i+1} = (G_i)_{T_{i+1}}$ (see Figure 6). If T' already has only one edge orbit then $T_{i+1} = T_i$ and $G_{i+1} = G_i$.

In both constructions, we pass to subgroups that split as graphs of groups such that the edge groups and vertex groups are \leq the edge and vertex groups of the original splitting of the overgroup.

We start with the amalgamated free product case. Let $T = T_0$ be the Bass-Serre tree corresponding to the splitting given in (1). Take the blow up $T_0^{(2)}$. If one of the vertex groups of this blow-up coincides with an incident edge group we are done. Otherwise we may pass to the $G^{(1)}$ tree $T_0^{(1)}$ which still has one edge orbit, two vertex orbits and whose vertex groups are one ended relative to the incident edge groups. Furthermore because the new vertex groups are \leq the vertex groups of T , if the statement of Theorem holds for $G^{(1)}$ and the splitting corresponding to its action on $T_0^{(1)}$ (also an amalgamated free product) then we are done.

We can now apply our second construction to the $G_0^{(1)}$ -tree $T_0^{(1)}$ to obtain a G_1 -tree T_1 , which again must have one edge orbit and two vertex orbits. Furthermore for the (conjugacy class) of the edge group, we have a proper containment $C_1 < C$. Again, because of the vertex groups of T_1 are \leq the vertex groups of $T_0^{(1)}$ if the Theorem holds for this subgroup, it holds for G .

We repeatedly apply our construction obtaining a sequence of groups that split as amalgamated free products, each time we do the second construction we pass to a smaller edge group; so that by accessibility, at some point there is some subgroup G_i acting on $T_i^{(2)}$ (see first construction) such that the vertex groups split as graphs of groups with finite edge groups and one of the incident edge groups coincides with the vertex group. So, since \leq is transitive, (1) of Theorem 1.6 is satisfied.

We now tackle the HNN extension case. The proof goes the same way, we repeatedly blow up, cleave, and pass to subtrees; the main difference is that the G -tree T has only one vertex orbit. If at some point one of the trees T_i or $T_i^{(1)}$ has two vertex orbits, then these vertex groups are vertex groups of a splitting

of the vertex group of T_{i-1} with finite edge groups. It therefore follows that if T_i satisfies (1) of Theorem 1.6, then T_{i-1} satisfies (2) of Theorem 1.6; thus so must our original splitting T , by transitivity of \leq . Otherwise the proof goes through identically. \square

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